Distinguishing Number of some Circulant Graphs

Sylvain Gravier\(^{(1)}\) & Kahina Meslem\(^{(2)}\) & Souad Slimani\(^{(3)}\)

UMR 5582 CNRS
\(^{(1)}\) Université Grenoble-Alpes, 100 rue des Maths, BP 74, 38402 St Martin d’Hères, France
\(^{(2,3)}\) LaROMad, Fac. Maths, USTHB, PO 32, 16111 Bab Ezzouar, Algeria

\(^{(1)}\) sylvain.gravier@ujf-grenoble.fr, \(^{(1)}\) kmeslem@usthb.dz, \(^{(1)}\) sslimani@usthb.dz

Abstract: Introduced by Albertson and al. [1], the distinguishing number \(D(G)\) of a graph \(G\) is the least integer \(r\) such that there is a \(r\)-labeling of the vertices of \(G\) that is not preserved by any nontrivial automorphism of \(G\). Most of graphs studied in literature have 2 as a distinguishing number value except complete, multipartite graphs or cartesian product of complete graphs not depend on \(n\). In this paper, we study circulant graphs of order \(n\) where the adjacency is defined using a symmetric subset \(A\) of \(\mathbb{Z}_n\), called generator. We show that we can built a family of circulant graphs of order \(n\) having distinct distinguishing numbers not depend on \(n\).

Key Words: Distinguishing Number, Automorphism of a Graph, Circulant Graph.

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Résumé : Introduit par Albertson et al. [1], le nombre distinguishing \(D(G)\) d’un graphe \(G\) est le plus petit nombre \(r\) tel que \(G\) admet une \(r\)-coloration de ses sommets, pas nécessairement propre, où celle-ci n’est pas préservée par aucun automorphisme non trivial de \(G\). Il a été démontré que la plupart des graphes \(G\) non rigides(i.e \(D(G) \neq 1\)) admettent un nombre distinguishing égale à deux excepté certains graphes tels que les complets, les multipartis complets et les produits cartésiens des complets. Le nombre distinguishing de ces graphes dépend de l’ordre de \(G\). Dans ce contexte, on détermine dans ce papier le nombre distinguishing des graphes circulants et on démontre que pour une séquence finie de nombres distinguishing, on peut trouver une famille de graphes circulants d’ordre commun ayant respectivement le nombre distinguishing de la séquence donnée.

Mots clés : Distinguishing Number, Automorphisme d’un graphe, Graphe circulant.
1 Introduction

In 1979, F. Rudin [12] proposed a problem in Journal of Recreational Mathematics by introducing the concept of the breaking symmetry in graphs. Albertson and al. [1] studied the distinguishing number in graphs defined as the minimum number of labels needed to assign to the vertex set of the graph in order to distinguish any non trivial automorphism graph. The distinguishing number is widely focused in the recent years: many articles deal with this invariant in particular classes of graphs: trees [3], hypercubes [2], product graphs [10] [9] [8] and interesting algebraic properties of distinguishing number were given in [11] [13] and [14]. Most of non rigid structures of graphs (i.e structures of graphs having at most one non trivial automorphism) need just two labels to destroy any non trivial automorphism. In fact, paths $P_n$ ($n > 1$), cycles $C_n$ ($n > 5$), hypercubes $Q_n$ ($n > 3$), $r$ ($r > 3$) times cartesian product of a graph $G'$ where $G$ is of order $n > 3$, circulant graphs of order $n$ generated by $\{\pm 1, \pm 2, \ldots \pm k\}$ [7] ($n \geq 2k + 3$) have $2$ as a common value of distinguishing number. However, complete graphs, complete multipartite graphs [4] and cartesian product of complete graphs (see [8] [5] [6]) are the few classes with a big distinguishing number. The associated invariant increases with the order of the graphs. In order to surround the structure of a graph of a given order $n$ and get a proper distinguishing number we built regular graphs $C(m,p)$ of order $mp$ where the adjacency is described by introducing a generator $A$ ($A \subset Z_{m,p}$). These graphs are generated by $A = \{(p - 1) + r.p,(p + 1) + r.p : 0 \leq r \leq m - 1\}$ for all $n = m.p \geq 3$. In fact, the motivation of this paper is to give an answer to this following question, noted (Q):

"Given a sequence of ordered and distinct integer numbers $d_1, d_2, \ldots, d_r$ in $\mathbb{N}^* \setminus \{1\}$, does it exist an integer $n$ and $r$ graphs $G_i$ ($1 \leq i \leq r$) such that $D(G_i) = d_i$ for all $i = 1, \ldots, r$ and $n$ is the common order of the $r$ graphs?"

In the following proposition, we give the answer to this question:

**Proposition 1** Given an ordered sequence of $r$ distinct integers $d_1, d_2, \ldots, d_r$ with $r \geq 2$ and $d_i \geq 2$ for $i = 1, \ldots, r$, there exists $r$ graphs $G_1, G_2, \ldots, G_r$ of order $n$ such that $G_i$ contains a clique $K_{d_i}$ and $D(G_i) = d_i$ for all $1 \leq i \leq r$.

**Preuve.** Suppose that $d_1 \neq 2$ and $n = d_r$. For the integer $d_r$, we assume that $G_r \simeq K_{d_r}$ and $D(G_r) = d_r$.

For the other integers, we consider the disconnected $(r - 1)$ graphs $G_i$ having two connected component $C$ and $C'$ such that $C \simeq K_{d_i}$ and $C'$ is a path $P_{n - d_i}$ for all $i = 1, \ldots, (r - 1)$.

Observe that, when $d_1 \neq 2$ or $n = d_r \neq 4$, then the connected component $C$ and $C'$ can not be isomorphic. By consequence, an automorphism $\delta$ of a graph $G_i$ acts in the same connected component for all $1 \leq i \leq r - 1$. More than, $D(G_i) = \max(D(C), D(C')) = D(C) = d_i$ for all $1 \leq i \leq r - 1$.

If $d_1 = 2$ and $n = d_r = 4$ the same graphs are considered except for $G_1$ where we put $G_1 \simeq P_4$. Then, $D(G_1) = 2 = d_1$.  

The graphs of Proposition 1 are not completely satisfying since these ones are not connected. Furthermore, these graphs give no additional information for graphs having hight distinguishing number, since they just use cliques for construction. So our purpose is to construct connected graphs structural properties that give answer to question (Q).

**Theorem 2** Given an ordered sequence of $r$ distinct integers $d_1, d_2, \ldots, d_r$ with $r \geq 2$ and $d_i \geq 2$ for $i = 1, \ldots, r$, there exists $r$ connected circulant graphs $G_1, G_2, \ldots, G_r$ of order $n$ such that $D(G_i) = d_i$. 


So, in section 1, basic definitions and preliminary results used in this paper are given. Then in section 2, we define circulant graphs $C(m,p), n = m.p \geq 3$ and provide interesting structural properties of this class of graphs. These later are used to determine the associated distinguishing number which is given in section 3. We also give the proof of theorem 2 in the same section. Finally, in section 4, we conclude by some remarks and possible improvement to reply of the question ($\mathcal{Q}$).

## 2 Definitions and Preliminaries Results

We only consider finite, simple, loopless, and undirected graphs $G = (V, E)$ where $V$ is the vertex set and $E$ is the edge set. The complement of $G$ is the simple graph $\overline{G} = (V, \overline{E})$ which consists of the same vertex set $V$ of $G.$ Two vertices $u$ and $v$ are adjacent in $\overline{G}$ if and only if they are not in $G.$ The neighborhood of a vertex $u$, denoted by $N(u)$, consists in all the vertices $v$ which are adjacent to $u.$ A complete graph of order $n$, denoted $K_n$, is a graph having $n$ vertices such that all two distinct vertices are adjacent. A complete bipartite graph is a bipartite graph where each vertex from any set adjacent to each vertex to another set. A path on $n$ vertices, denoted $P_n$, is a sequence of distinct vertices and and $n-1$ edges $v_1v_i+1, 1 \leq i \leq n-1$. A path relying two distinct vertices $u$ and $v$ in $G$ is said $uv$-path. A cycle, on $n$ vertices denoted $C_n$, is a path with $n$ distinct vertices $v_1, v_2, \ldots, v_n$ where $v_1$ and $v_n$ are confused. For a graph $G$, the distance $d_G(u, v)$ between vertices $u$ and $v$ is defined as the number of edges on a shortest $uv$-path.

Given a subset $A \subset \mathbb{Z}_n$ with $0 \notin A$ and for all $a \in A$ and $-a \in A$. A circulant graph, is a graph on $n$ vertices $\{0,1, \ldots, n-1\}$ where two vertices $i$ and $j$ are adjacent if $j - i$ modulo $n$ is in $A$.

The automorphism (or symmetry) of a graph $G = (V, E)$ is a permutation $\sigma$ of the vertices of $G$ preserving adjacency i.e if $xy \in E$, then $\sigma(x)\sigma(y) \in E$. The set of all automorphisms of $G$, noted $\text{Aut}(G)$ defines a structure of a group. A labeling of vertices of a graph $G$, $c : V(G) \to \{1, 2, \ldots, r\}$ is said $r$-distinguishing of $G$ if $\forall \sigma \in \text{Aut}(G) \setminus \{\text{id}_G\}: c \neq c \circ \sigma$. That means that for each automorphism $\sigma \neq \text{id}$ there exists a vertex $v \in V$ such that $c(v) \neq c(\sigma(v))$. A distinguishing number of a graph $G$, denoted by $D(G)$, is a smallest $r$ such that $G$ is an $r$-distinguishing labeling. Since $\text{Aut}(G) = \text{Aut}(\overline{G})$, we have $D(G) = D(\overline{G})$. The distinguishing number of a complete graph of order $n$ is equal to $n$. The distinguishing number of complete multipartite graphs is given in the following theorem:

**Theorem 3** \cite{4} Let $K_{a_1^{a_1}, a_2^{a_2}, \ldots, a_r^{a_r}}$ denote the complete multipartite graph that has $j_i$ partite sets of size $a_i$ for $i = 1, 2, \ldots, r$ and $a_1 > a_2 > \cdots > a_r$. Then $D(K_{a_1^{a_1}, a_2^{a_2}, \ldots, a_r^{a_r}}) = \min\{p : \left(\frac{p}{a_i}\right) \geq j_i \text{ for all } i\}$

Let us introduce the concept of modules useful to investigate distinguishing number in graphs. A module in the graph $G$ is a subset $M$ of vertices which share the same neighborhood outside $M$ i.e for all $y \in V \setminus M: M \subseteq N(y)$ or $xy \notin E$ for all $x \in M$. A trivial module in a graph $G$ is either the set $V$ or any singleton vertex. A module $M$ of $G$ is said maximal in $G$ if for each non-trivial module $M'$ in $G$ containing $M$, $M'$ is reduced to $M$. The following lemma shows how modules can help us to estimate the value of distinguishing number in graphs:

**Lemma 4** Let $G$ be a graph and $M$ a module of $G$. Then, $D(G) \geq D(M)$
Preuve. Let \( c \) be an \( r \)-labeling such that \( r < D(M) \). Since \( r < D(M) \), there exits \( \delta \mid_M \) a non trivial automorphism of \( M \) such that \( c(x) = c(\delta \mid_M (x)) \) for all \( x \in M \) i.e the restriction of \( c \) in \( M \) is not a distinguishing. Now, let \( \delta \) be the extension of \( \delta \mid_M \) to \( G \) with \( \delta(x) = x \) \( \forall x \notin M \) and \( \delta(x) = \delta \mid_M (x) \) otherwise. We get \( c(x) = c(\delta(x)) \) for all \( x \in G \). Moreover, \( \delta \neq \text{id} \) since \( \delta \mid_M \neq \text{id} \mid_M \).

3 Circulant Graphs \( C(m, p) \)

In this section, we study distinguishing number of circulant graphs \( C(m, p) \) of order \( n = mp \geq 3 \) with \( m \geq 1 \) and \( p \geq 2 \). A vertex \( i \) is adjacent to \( j \) in \( C(m, p) \) iff \( j - i \) modulo \( n \) belongs to \( A = \{p-1+r.p, p+1+r.p, 0 \leq r \leq m-1\} \) (See Fig. 1), where \( p > 1 \). These graphs are circulant since for all \( 0 \leq r \leq m-1 \), the symmetric of \( p-1+r.p \) is \( 1+r.p+(-m-r-2)p \) which belongs to \( A \). We have \( 0 \notin A \) since \( p > 1 \). Let specify some other particular values of \( p \) and \( m \), \( C(m, 1) \) is the clique \( K_m \) and \( C(1, p) \) is the cycle \( C_p \). Also we have: \( C(m, 2) = K_{m,m} \) and \( C(m, 3) = K_{m,m,m} \).

By theorem 3, \( D(C(m, 2)) = D(C(m, 3)) = m + 1 \). Moreover, \( D(C(1, p)) = 2 \) for \( p \geq 6 \).

Property 3.1 The vertex set of \( C(m, p) \) \((m \geq 2 \text{ and } p \geq 2)\) can be partitioned into \( p \) stable modules \( M_i = \{i + r.p : 0 \leq r \leq m-1\} \) of size \( m \) for \( i = 0, \ldots, p - 1 \).

Preuve. Given two distinct vertices \( a, b \in M_i \) for \( i = 0, \ldots, p - 1 \), \( a - b \equiv rp[n] \) for some \( 0 < r \leq m - 1 \), then \( a - b \notin A \) which proves that each \( M_i \) induces a stable sets.

Moreover, it is clear that \( \{M_i\}_{i=0,\ldots,p-1} \) forms a partition of vertex set of \( C(m, p) \).

Let us prove that \( M_i \) defines a module. For this, suppose that \( a = i + r_a \cdot p \) and \( b = i + r_b \cdot p \) two distinct vertices of a given stable set \( M_i \).

Let \( c \in V \setminus M_i \) such that \( ac \) is an edge and let \( c = j + r_c \cdot p \).

Let
\[
r_{bc} = \begin{cases} r_b - r_c & \text{if } rb > rc \\ m + (rb - rc) & \text{else} \end{cases}
\]
\[
r_{ac} = \begin{cases} r_a - r_c & \text{if } ra > rc \\ m + (ra - rc) & \text{else} \end{cases}
\]

two integer numbers such that \( b - c \equiv (i - j) + r_{bc} \cdot p[n] \) and \( a - c \equiv (i - j) + r_{ac} \cdot p[n] \) (with \( 0 \leq r_{ac} \leq m - 1 \) and \( 0 \leq r_{bc} \leq m - 1 \)).

Since \( a - c \) is in \( A \) then there is some integers \( k \) verifying \( 0 \leq k \leq r_{ac} \) such that \( i - j + kp = p - 1 \) (or \( p + 1 \)).

If \( k \leq r_{bc} \), we obtain \( b - c \equiv i - j + kp + (r_{bc} - k) \cdot p[n] \).

Then \( b - c \equiv p - 1 + (r_{bc} - k) \cdot p[n] \) (or \( p + 1 + (r_{bc} - k) \cdot p[n] \)). We deduce that \( b - c \in A \) since \( 0 \leq k \leq m - 1 \).

Else, we have \( r_{bc} < k \leq m + r_{bc} \). We have \( b - c \equiv i - j + r_{bc} \cdot p[n] \). Then \( b - c \equiv i - j + (m + r_{bc}) \cdot p[n] \).

We get \( b - c \equiv i - j + kp + (m + r_{bc} - k) \cdot p[n] \) which belongs to \( A \) since \( 0 \leq m + r_{bc} - k \leq m - 1 \).

Since each \( M_i \) (for all \( 0 \leq i \leq p - 1 \)) is a stable set then, by definition of a module, we have:

Property 3.2 Any permutation of elements of \( M_i \) is an automorphism of \( G \) for all \( 0 \leq i \leq p - 1 \).

By lemma 4 and property 3.1, we have \( D(C(m, p)) \geq m \). We will improve this bound:

Theorem 5 For all \( p \geq 2 \) and for all \( m \geq 2 \), \( D(C(m, p)) = m + 1 \) if \( p \neq 4 \).
4 Proof of Theorem 2 and Theorem 5

In this section, we give the proof of theorem 5 in the first step, while the second step is spent to give the proof of the theorem 2.

Lemma 6 For all \( p \geq 2 \) and for all \( m \geq 2 \), \( D(C(m, p)) > m \).

Proof. If \( p = 2 \) (resp. \( p = 3 \)) then \( C(m, 2) \cong K_{m,m} \) (resp. \( C(m, 3) \cong K_{m,m,m} \)). According to Theorem 3, we have \( D(C(m, p)) > m \). Let \( C(m, p) \) be the circulant graph generated by \( A = \{p - 1 + rp, p + 1 + rp : 0 \leq r \leq m - 1\} \).

Let us suppose that \( p > 3 \). Since the modules \( M_i \) (\( i = 0, \ldots, p - 1 \)) are stables of size \( m \), then by lemma 4 we have \( D(C(m, p)) \geq m \).

Consider \( c : V(C(m, p)) \to \{1, 2, \ldots, m\} \) be a \( m \)-labeling of \( C(m, p) \) (\( m \geq 2 \)) and prove that \( c \) is not \( m \)-distinguishing.

By way of contradiction, assume that \( c \) is \( m \)-distinguishing.

For all distinct vertices \( v, w \) in a given module \( M_{i_0} \) with \( i_0 \in \{0, 1, \ldots, p-1\} \) we have \( c(v) \neq c(w) \) otherwise, there exists a transposition \( \tau \) of \( v \) and \( w \) verifying \( c = c \circ \tau \). This yields a contradiction.

That means that in a fixed module \( M_i \) we have all labels.

Let \( P_j \) (\( 1 \leq j \leq m \)) be a set of index \( \{(j-1)p + i, i \in \{0, \ldots, p-1\}\} \).

Let \( v \in M_i \) (\( 0 \leq i \leq p - 1 \)) then \( v = i + rp \) where \( 0 \leq r \leq m - 1 \). Consider now the mapping \( \delta_i \) with \( i = 0, \ldots, p - 1 \) defined as follows: \( \delta_i : V \to V \) such that \( \delta_i(v) = (c(v) - 1)p + i \) if \( v \in M_i \) else \( \delta(v) = v \). By property 3.2, \( \delta_i \) defines an automorphism of \( G \).

Let \( \delta = \delta_0 \circ \cdots \circ \delta_{p-1} \) be an automorphism of \( G \).

Let \( \psi \) be a mapping defined as follows: \( \psi : V \to V \) such that \( \psi(i + rp) = p - (i + 1) + rp \). Let prove that \( \psi \) is an automorphism of \( G \).

Let \( a = i + rp \) and \( b = j + r'p \) two adjacent vertices then \( b - a = j - i + (r' - r)p \in A \). We have \( \psi(b) - \psi(a) = i - j + (r' - r)p \) which belongs to \( A \). Thus \( \psi \) is an automorphism of \( G \).

Check now that \( \delta^{-1} \circ \psi \circ \delta \) is non trivial automorphism of \( G \) preserving the labeling \( c \) (See Fig.2).

Then \( \delta^{-1} \circ \psi \circ \delta \) is clearly an automorphism because it is a composition of automorphisms.
Figure 2: The automorphism $\delta^{-1} \circ \psi \circ \delta$ applied to $C(4, 4)$ with four labels (1, 2, 3, 4) = (black, red, blue, green).

Since $\delta^{-1} \circ \psi \circ \delta(0) = \delta^{-1} \circ \psi((c(0) - 1)p + 0) = \delta^{-1}((c(0) - 1)p + (p - 1)) = u$ with $u \in M_{p-1}$ and $c(u) = c(0)$, then $u \neq 0$ since $0 \in M_0$ and $M_0 \neq M_{p-1}$ and $p > 1$. Thus $\delta^{-1} \circ \psi \circ \delta$ is not a trivial automorphism.

To complete the proof, it is enough to show that $c(u) = c(\delta^{-1} \circ \psi \circ \delta(u))$ for all $u$.

Let $u = i + rp$ then we have $\delta^{-1} \circ \psi \circ \delta(u) = \delta^{-1} \circ \psi((c(u) - 1)p + i) = \delta^{-1}((c(u) - 1)p + p - (i + 1)) = v$ such that $v \in M_{p-(i+1)}$ and $c(v) = c(u)$.

Then $\delta^{-1} \circ \psi \circ \delta$ preserves the labeling. \qed

The following result gives the exact value of $D(C(m, p))$

Lemma 7 For all $p \geq 2$ and $p \neq 4$ and for all $m \geq 2$ : $D(C(m, p)) \leq m + 1$

Preuve. If $p \in \{2, 3\}$ the proposition is true by theorem 3. Consider $c$ be the $(m + 1)$-labeling defined as follows (See Fig. 3):
Figure 3: The \((m+1)\)-labeling: the label of each vertex is given inside the cycle.

\[
c(v) = \begin{cases} 
1 & 0 \leq v \leq \left\lfloor \frac{p}{2} \right\rfloor \text{ and } v = 2p - 1 \\
2 & \left\lfloor \frac{p}{2} \right\rfloor < v \leq p - 1 \\
j + 1 & v \in P_j \text{ and } 2 \leq j \leq m \text{ and } v \neq 2p - 1
\end{cases}
\]

Suppose that there exists an automorphism \(\delta\) preserving this labeling and prove that \(\delta\) is trivial.

Since \(p > 4\), 0 is the unique vertex labeled 1 which has the following sequence of label in his neighborhood \((1, 1, 2, 3, 4, 4, \ldots, m+1, m+1)\). Thus \(\delta(0) = 0\).

However, we refer to the following claim:

**Claim 1** For each vertex \(i\) in \(C(m, p)\) where \(0 \leq i \leq p - 1\), we have:

\[
d(0, i) = \begin{cases} 
i & 1 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor \\
p - i & \left\lfloor \frac{p}{2} \right\rfloor < i \leq p - 1
\end{cases}
\]

**Preuve.** First observe that for all pair of vertices \(u\) and \(v\) in the same module \(M\) and \(z \in V \setminus M\), we have \(d(u, z) = d(v, z)\) and \(d(u, v) = 2\).

Now, if we contract each module \(M_i\) of \(C(m, p)\), then we get a cycle on \(p\) vertices which implies the claim.

Let us prove that each vertex labeled 1, is fixed by the automorphism \(\delta\):

Consider the table describing the sequence of labels of the vertex \(u\):
Additionally, observe that any vertex \( i \) is \((1,1,3,3,\ldots m+1,m+1)\). More than, for all two distinct vertices \( u \) and \( v \) such that \( 0 < u,v < \left\lfloor \frac{p}{2} \right\rfloor \) we have \( d(u,0) \neq d(v,0) \). Then, since \( \delta(0) = 0 \) we get \( \delta(u) = u \) and \( \delta(v) = v \). Generally, for each vertex \( i \) such that \( 0 < i < \left\lfloor \frac{p}{2} \right\rfloor \), we obtain \( \delta(i) = i \).

More than, the sequence of labels in the neighborhood of \( 2p-1 \) and \( \left\lfloor \frac{p}{2} \right\rfloor \) is \( \{1,2,3,3,4,4,\ldots ,m+1, m+1\} \). Since \( d(\left\lfloor \frac{p}{2} \right\rfloor ,0) > d(2p-1,0) = 1 \), then we get \( \delta(2p-1) = 2p-1 \) and \( \delta(\left\lfloor \frac{p}{2} \right\rfloor ) = \left\lfloor \frac{p}{2} \right\rfloor \).

Now observe that by Claim 1, any distinct vertices \( u \) and \( v \) labeled 2, we have \( d(u,0) \neq d(v,0) \). Then for any vertex \( u \) such that \( c(u) = 2 \), we have \( \delta(u) = u \).

Finally, let us prove that each vertex \( v \) in \( C(m,p) \setminus (P_1 \cup \{2p-1\}) \) is fixed by the automorphism \( \delta \). For that, it is enough to show for all pair of distinct vertices \( u \) and \( v \) such that \( c(u) = c(v) \), we have \( N(u) \cap \{0,1,2,\ldots ,p-1\} \neq N(v) \cap \{0,1,2,\ldots ,p-1\} \). This proposition will imply that each vertex \( v \) labeled \( c(v) \) (\( c(v) \geq 2 \)) is fixed by \( \delta \) and we conclude the proof of theorem.

Let \( u \) and \( v \) two distinct vertices such that \( c(u) = c(v) \) with \( u,v \in C(m,p) \setminus (P_1 \cup \{2p-1\}) \).

Since \( c(u) = c(v) \), we have \( u \in M_i \) and \( v \in M_j \) with \( i \neq j \). Then \( i-1, i+1 \in N(u) \) and \( j-1, j+1 \in N(v) \).

If \( i = 0 \) then \( p-1 \in N(u) \) since \( p \in M_i \). Similarly, if \( i = p-1 \), then \( 0 \in N(u) \) since \( mp-1 \in M_i \).

Therefore, modulo \( p \), we have that \( i-1, i+1 \in N(u) \cap \{0,1,\ldots ,p-1\} \) and \( j-1, j+1 \in N(v) \cap \{0,1,\ldots ,p-1\} \).

Additionally, observe that any vertex \( u \) has exactly two neighborhood among \( p \) consecutive vertices of \( G \). Thus \( N(u) \cap \{0,1,\ldots ,p-1\} = \{i-1,i+1 \mod p\} \) and \( N(v) \cap \{0,1,\ldots ,p-1\} = \{j-1,j+1 \mod p\} \).

Now, if \( N(u) \cap \{0,1,\ldots ,p-1\} = N(v) \cap \{0,1,\ldots ,p-1\} \) and \( i \neq j \), then \( i+1 = j-1 \) and \( i-1 = j+1 \). Thus \( j = i-2 \), \( j = i+2 \) and \( p = 4 \).

Since \( p > 4 \), we get that \( N(u) \cap \{0,1,\ldots ,p-1\} \neq N(v) \cap \{0,1,\ldots ,p-1\} \). ■

Lemma 6 and lemma 7 give the proof of theorem 5. The following result gives the value of distinguishing number for \( p = 4 \):

**Corollary 8** For each \( m \geq 2 \), \( C(m,4) \) is isomorphic to \( C(2m,2) \) (or \( K_{2m,2m} \)) and \( D(C(m,4)) = 2m+1 \).

**Preuve.** The graph \( C(m,4) \) is partitioned into four modules \( M_0, M_1, M_2, M_3 \). We have: \( N(M_0) = N(M_2) = M_1 \cup M_3 \) and \( N(M_1) = N(M_3) = M_0 \cup M_2 \). Thus, the module \( M_i \) is not maximal where \( i \in \{0,1,2,3\} \). Furthermore, \( M_0 \cup M_2 \) and \( M_1 \cup M_3 \) are stables of size.
Then, the graph \( C(m, 4) \) is a multipartite graph \( K_{2m,2m} \) and \( D(C(m, 4)) = D(K_{2m,2m}) = D(C(2m, 2)) = 2m + 1. \)

**Proof of Theorem 2**

Let \( d_1, d_2, \ldots, d_r \) be an ordered sequence of distinct integers. Let \( m_i = d_i - 1 \) for all \( i = 1, \ldots, r \) and \( p_i = \prod_{j \neq i} m_j \).

By definition, \( m_i p_i = m_j p_j \) for \( i \neq j \) for \( i, j = 1, \ldots, r \).

If all \( p_i \neq 4 \), then let \( n = m_i p_i \) else \( n = 3m_i p_i \) for all \( i = 1, \ldots, r \).

Now, by Theorem 5, \( D(C(m_i, p_i)) = m_i + 1 = d_i \) for all \( i = 1, \ldots, r \).

So, \( (G_i)_i = (C(m_i, p_i))_i \) with \( i = 1, \ldots, r \), is a family of connected circulant graphs of order \( n \) such that \( D(G_i) = d_i \).

5 Remarks and conclusion

We have studied the structure of circulant graphs \( C(m, p) \) by providing the associated distinguishing number. We have determined the distinguishing number of circulant graphs \( C(m, p) \) for all \( m, p \geq 3 \) with \( m \geq 1 \) and \( p \geq 2 \). We can summarize the result which give the value of distinguishing number for circulant graphs \( C(m, p) \) as follows:

\[
D(C(m, p)) = \begin{cases} 
  m & (m \geq 3 \text{ and } p = 1) \\
  m + 1 & (m = 1 \text{ and } p \geq 6) \text{ or } (m \geq 2 \text{ and } p \geq 2 \text{ and } p \neq 4) \\
  2m + 1 & (m = 1 \text{ and } p \in \{3, 4, 5\}) \text{ or } (m \geq 2 \text{ and } p = 4) 
\end{cases}
\]

We deduce that for a given integer \( n = \prod_{i=1}^{r} m_i \) for \( r \geq 2 \) and \( m_i \geq 1 \), we can build a family of graphs of same order \( n \) where the distinguishing number depends on divisors of \( n \). The main idea of constructing such graphs consists of partitioning the vertex set into modules of same size. The circulant graphs are well privileging structure. One may ask if we can construct such family of circulant graphs with smaller order?

For instance, we can improve in Theorem 2 the order \( n \) of \( (C(m_i, p_i))_i \) for \( i = 1, \ldots, r \), by taking

\[
n = \frac{\prod_{i=1}^{r} m_i}{\gcd(m_i, \prod_{j<i} m_j)}.
\]

**References**


